Malaysian Journal of Mathematical Sciences 19(3): 1019–1034 (2025)
https://doi.org/10.47836/mjms.19.3.13

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

Characteristics of the Enhanced Power Coprime Graphs of Some Finite Groups

Mohamed, N.*[1,2], Mohd Ali, N. M.[1], and Bello, M.[3]

[1]*Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia*
[2]*Mathematical Sciences Studies, College of Computing, Informatics and Mathematics, Universiti Teknologi MARA Cawangan Negeri Sembilan Kampus Seremban, 70300 Seremban, Negeri Sembilan, Malaysia*
[3]*Department of Mathematics and Statistics, Federal University of Kashere, P.M.B 0182 Gombe, Gombe State, Nigeria*

E-mail: norlyda1618@uitm.edu.my
**Corresponding author*

Received: 26 July 2024
Accepted: 10 April 2025

Abstract

The graphs of groups is a powerful tool for studying finite groups as it visually represents their structures and relationships. In general, introducing new graphs of groups leads to discoveries about the graphs' characteristics, offering significant insights into its structure, connectivity, and spectral aspects. This paper investigates various characteristics of a newly introduced graph called the enhanced power coprime graph of some finite groups. The enhanced power coprime graph of a finite group, G is defined as a graph with elements of G as vertices, where two distinct vertices x and y are adjacent if they generate a proper cyclic subgroup of G and $\gcd(|x|, |y|) = 1$. First, we establish the general presentations of the enhanced power coprime graph for all semidihedral groups of order 2^n and prime power cases of dihedral and generalized quaternion groups. These presentations facilitate the determination of various characteristics, including vertex degrees, clique numbers, chromatic numbers, independence numbers, domination numbers, girth, diameter, graph classification, and the Laplacian spectrum. The results reveal that the enhanced power coprime graphs of the mentioned groups are connected, planar, and perfect, with consistent characteristics for those with similar presentations. These findings have applications in computational group theory, network analysis, and coding theory, using graph characteristics to explore group structures.

Keywords: enhanced power coprime graph; enhanced power graph; coprime graph; graph invariants; Laplacian spectrum.

1 Introduction

The study of finite groups, central to understanding symmetry and transformation, has been a fundamental aspect of algebra for over a century [10]. The graphs of finite groups reveal complex interactions within group structures. Their characteristics, such as invariants, classifications, and spectral properties, provide deep insights into group connectivity. Researchers have leveraged these graph presentations to uncover properties with applications in computational group theory, network analysis, and coding theory. Numerous studies have explored various types of graphs of groups, presenting group theory through the lens of graph theory. For instance, Bera [6] introduced the intersection power graph, suggesting that new graphs can be defined on algebraic structures by studying the poset of suitable substructures.

Recently, Bello et al. [5] introduced the commuting order product prime graph, which is a variation of the commuting graph and order product prime graph. After that, Cameron and Kuzma [12] introduced the deep commuting graph, equivalent to a subgraph of the enhanced power graph and the transitive closure of the commuting graph. Mohamed et al. [19] introduced the cyclic order product prime graph to study the variation between cyclic graph and the order product prime graph. These developments have sparked interest in new intersection graphs, particularly enhanced power and coprime graphs. Let G be a finite group with identity e. The enhanced power graph, $\Gamma^{ep}(G)$, is a graph where two distinct vertices x and y are adjacent if $\langle x, y\rangle$ is a cyclic subgroup of G. It is important to note that $\Gamma^{ep}(G)$ lies between the power graph and the commuting graph [1]. Additionally, $\Gamma^{ep}(G)$ can also be derived from the directed power graph by connecting two vertices if they both exist in the closed-out neighborhood of a certain vertex [12]. In contrast, the coprime graph, $\Gamma^{c}(G)$, captures relationships between elements whose orders are relatively prime, where two distinct vertices x and y are connected by an edge if and only if $\gcd(|x|, |y|) = 1$ [14].

Considerable attention has been given to the study of the characteristics of enhanced power graphs. For example, Bera and Bhuniya [7] classified the enhanced power graphs into three distinct categories: complete graphs, Eulerian graphs, and graphs that can be dominated under certain conditions. Furthermore, Panda et al. [21] and Dalal and Kumar [13] have studied various graph invariants of these graphs, including the chromatic number, independence number, minimum degree, matching number, metric dimension, and covering number, focusing on specific groups selected for analysis. In addition, Bera et al. [8] examined the connectivity of vertices in $\Gamma^{ep}(G)$. Related surveys and open problems concerning this graph have also been addressed by Ma et al. [18] and Dupont et al. [15]. Recently, Parveen et al. [22] investigated the Laplacian spectrum of this graph, presenting results for semi-dihedral, dihedral, and generalized quaternion groups.

Apart from that, Ma et al. [17] explored the coprime graph and analyzed how its theoretical features influence the properties of the group. Dorbidi [14] classified all groups for which $\Gamma^{c}(G)$ is a complete r-partite graph or a planar graph. The basic properties of $\Gamma^{c}(G)$ were also discussed by Selvakumar and Subajini [26]. Banerjee [3] investigated the Laplacian spectrum of $\Gamma^{c}(G)$ for finite cyclic and dihedral groups. Sehgal et al. [25] and Adhikari and Banerjee [2] extended the study to coprime order graphs (an extension of coprime graphs), determining the Laplacian spectra for certain cases of finite groups. Recently, Nurhabibah et al. [20] discovered numerical invariants of the coprime graph for generalized quaternion groups. A comparison of the aforementioned graphs, focusing on their distinctive characteristics that provide crucial insights into their unique aspects, can be found in Cameron [11].

Despite extensive research on the enhanced power graph and coprime graph, their combined structural insights remain largely unexplored. This paper introduces the enhanced power coprime graph, which integrates properties of both to analyze group structures with specific generating patterns and ordering properties. By focusing on the semi-dihedral groups of order 2^n and prime power cases of dihedral and generalized quaternion groups, we establish their general presentations, which are crucial for understanding graph connectivity. These presentations support structural analysis, as demonstrated in the commuting decompositions by Bhat and Sudhakara [9] and the connectivity classifications by Bello et al. [4], forming a foundation for proving key graph properties.

2 Background of Theory and Preliminaries

This section provides the background of the related theory and some preliminary research. The focus of this research is on certain finite groups G, including the dihedral groups of order $2n$,

$$D_n = \langle a, b \mid a^n = b^2 = (ab)^2 = e, \quad bab^{-1} = a^{-1} \rangle, \qquad \text{where} \qquad n \geq 3,$$

the semi-dihedral groups of order 2^n,

$$SD_{2^n} = \langle a, b \mid a^{2^{n-1}} = b^2 = e, \quad ba = a^{2^{n-2}-1} b \rangle, \qquad \text{where} \qquad n \geq 4,$$

and the generalized quaternion groups of order $4n$,

$$Q_{4n} = \langle a, b \mid a^{2n} = b^4 = e, \quad bab^{-1} = a^{-1}, \quad a^n = b^2 \rangle, \qquad \text{where} \qquad n \geq 2.$$

Here, e refers to the identity element of G, and $|a|$ denotes the order of an element a. Note that in these groups, $\langle a \rangle$ is isomorphic to a cyclic group. Thus, we can represent these groups using their cyclic subgroups and other additional subsets of G as follows,

1. $D_n = \langle a \rangle \cup \{a^i b \mid 0 \leq i < n\}$, where $\langle a \rangle = \{e, a, \ldots, a^{n-1}\}$.
2. $SD_{2^n} = \langle a \rangle \cup \{a^{2j} b \mid 0 \leq j < 2^{n-2}\} \cup \{a^{2j+1} b \mid 0 \leq j < 2^{n-2}\}$, where $\langle a \rangle = \{e, a, \ldots, a^{2^{n-1}-1}\}$.
3. $Q_{4n} = \langle a \rangle \cup \{a^k b \mid 0 \leq k < 2n\}$, where $\langle a \rangle = \{e, a, \ldots, a^{2n-1}\}$.

In this study, all graphs are simple and undirected. A graph Γ is represented by the sets of vertices V and edges E. The relationship between two vertices v_1 and v_2 in Γ is indicated by $v_1 \sim v_2$. A leaf vertex is a vertex with degree 1, which is connected to exactly one other vertex in Γ. A star graph is a tree with one central vertex connected to all other vertices, which are leaf vertices. The graph Γ is connected if there is a path between any two vertices, and a complete graph K_n is one in which every pair of vertices is connected by an edge. The graph Γ is regular if all vertices have the same degree. A graph Γ is planar if it can be represented on a two-dimensional plane without any edges intersecting, except at the vertices where they are incident. A graph Γ is perfect if the chromatic number of every induced subgraph equals the size of its largest clique. For subgraphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$, the disjoint union $\Gamma_1 \cup \Gamma_2$ forms a graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$, where V_1 and V_2 are disjoint. The join of Γ_1 and Γ_2, denoted as $\Gamma_1 + \Gamma_2$, is obtained by adding edges between all vertices of Γ_1 and Γ_2.

Graph invariants are essential for analyzing graph structures, including vertex degree, clique number, chromatic number, independence number, domination number, girth, and diameter. In a graph Γ, the degree of a vertex v, deg(v), represents the number of edges incident to it. The clique number, $\omega(\Gamma)$, is the size of the largest complete subgraph in Γ. The chromatic number,

$\chi(\Gamma)$, signifies the minimum number of colors required to color Γ so that no two adjacent vertices share the same color. The independence number, $\alpha(\Gamma)$, is the size of the largest set of mutually non-adjacent vertices, while the domination number, $\gamma(\Gamma)$, is the size of the smallest set of vertices such that every other vertex is adjacent to at least one vertex in this set. The girth is the length of the shortest cycle in Γ; if Γ has no cycles, its girth is infinite. The eccentricity of a vertex v, $\text{ecc}(v)$, is the greatest distance from v to any other vertex in Γ. The diameter of Γ, $\text{diam}(\Gamma)$, is the maximum eccentricity among all vertices in Γ. All the graph information given can be found in [16, 27]. Throughout this paper, we assume $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in G$.

The Laplacian matrix of Γ, $L(\Gamma)$, is a symmetric and positive semi-definite matrix representation of Γ with real and non-negative eigenvalues. It is defined as $L(\Gamma) = D(\Gamma) - A(\Gamma)$, where $D(\Gamma)$ is a diagonal matrix containing the degrees of Γ, and $A(\Gamma)$ is the adjacency matrix of Γ with entries such that if the vertices v_i and v_j are adjacent, the entry is 1; otherwise, it is 0. An interesting property of $L(\Gamma)$ is that the sum of its rows (or columns) is always zero, which means the matrix is singular. As a result, its smallest eigenvalue is always 0. The characteristic polynomial of $L(\Gamma)$, $\Theta(L(\Gamma))$, is defined as $\det(\rho I - L(\Gamma))$, where ρ is a scalar and I is the identity matrix. This polynomial captures the Laplacian spectrum of Γ, $L_{spec}(\Gamma)$, which provides insights into the algebraic and topological properties of the underlying groups. Listing the unique Laplacian eigenvalues of Γ in descending order as $\rho_{n_1}(\Gamma) \geq \rho_{n_2}(\Gamma) \geq \cdots \geq \rho_{n_r}(\Gamma) = 0$, with their respective multiplicities $m_1, m_2, \ldots, m_r$, the Laplacian spectrum is given by,

$$L_{spec}(\Gamma) = \{(\rho_{n_1}(\Gamma))^{m_1}, (\rho_{n_2}(\Gamma))^{m_2}, \ldots, (\rho_{n_r}(\Gamma))^{m_r}\}. \tag{1}$$

Related spectral analysis by Romdhini et al. [23] using the neighbors degree sum matrix highlights how eigenvalue-based methods can reveal structural properties of group-based graphs.

Some preliminary results that provide important information on the connectivity and adjacency of the related graphs are listed below.

Lemma 2.1. [7] *Let $c, d \in G$, and let $Gen(c)$ and $Gen(d)$ be the sets of all generators of the cyclic subgroups $\langle c\rangle$ and $\langle d\rangle$ of G, respectively. If $|c| = |d|$ and $\langle c\rangle \neq \langle d\rangle$, then x is not adjacent to y for all $x \in Gen(c)$ and $y \in Gen(d)$.*

Remark 2.1. *Let G be a finite group. If e is the identity element of G, then for all $y \in G \setminus \{e\}$,*

$$\langle e, y\rangle = \langle y, e\rangle = \langle y\rangle.$$

Theorem 2.1. [24] *Let SD_{2^n} be the semi-dihedral group of order 2^n. For any integer j with $0 \leq j < 2^{n-1}$,*

1. $|a^j| = \dfrac{2^{n-1}}{gcd(j, 2^{n-1})}$,
2. $|a^j b| = 2$ *when j is even, and* $|a^j b| = 4$ *when j is odd.*

Theorem 2.2. [26] *Let G be a finite group. Then $\Gamma^c(G)$ is a tree if and only if G is isomorphic to a p-group.*

Proposition 2.1. [19] *Let SD_{2^n} be the semi-dihedral group of order 2^n. For all $x, y \in G$, $\langle x, y\rangle$ is equal to a proper cyclic subgroup of G if,*

1. $x, y \in \langle a\rangle$,
2. $x = e$ *and* $y = a^j b$, *where* $0 \leq j < 2^{n-1}$,
3. $x = a^{2^{n-2}}$ *and* $y = a^{2j+1}b$, *where* $0 \leq j < 2^{n-2}$,
4. $x = a^{2j+1}b$ *and* $y = a^{j+2^{n-2}}b$, *where* $0 \leq j < 2^{n-2}$.

3 Main Results

This section introduces the enhanced power coprime graphs and establishes general presentations for certain finite groups. It also provides essential information about various graph invariants, including vertex degrees, clique numbers, chromatic numbers, independence numbers, domination numbers, girth, diameter, classifications of the defined graphs of groups, and findings related to the Laplacian spectrum. These details are derived from the established general presentations. The results comprehensively describe the structure of the enhanced power coprime graph, offering potential new concepts in group theory and graph theory.

3.1 General presentations of the defined graphs of groups

Before proving the general presentations, the enhanced power coprime graph of groups is introduced, along with an example of the visualization of the defined graph.

Definition 3.1. *Let G be a finite group with identity e, and let x and y be two distinct vertices of G. The enhanced power coprime graph, $\Gamma^{epc}(G)$, is the graph where x and y are adjacent if and only if $\langle x, y\rangle$ is a proper cyclic subgroup of G and $gcd(|x|, |y|) = 1$.*

Example 3.1. *Let G be either the dihedral group of order 8, D_4, or the semidihedral group of order 16, SD_{16}. In $\Gamma^{epc}(G)$, adjacency occurs only between the identity element e and $y \in G \setminus \{e\}$, as $\langle e, y\rangle = \langle y\rangle$ and $gcd(|e|, |y|) = 1$. Figures 1 and 2 illustrate these graphs for D_4 and SD_{16}, respectively.*

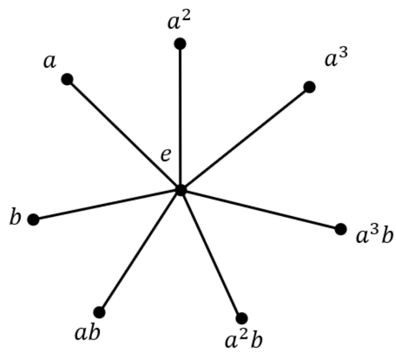

Figure 1: The enhanced power coprime graph of D_4.

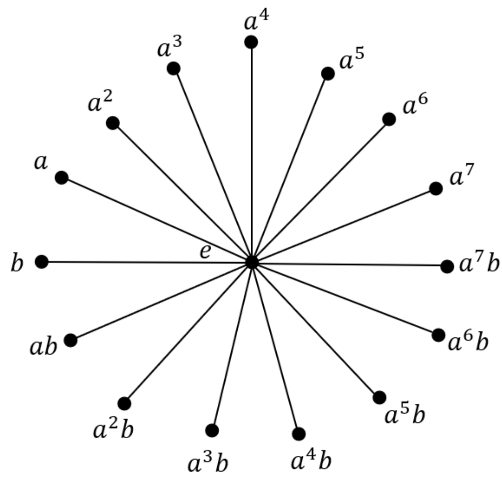

Figure 2: The enhanced power coprime graph of SD_{16}.

Example 3.2. *Let G be the generalized quaternion group of order 12, $Q_{12} = \langle a \rangle \cup \{a^k b \mid 0 \leq k < 6\}$, where $\langle a \rangle = \{e, a, a^2, \ldots, a^5\}$. For all $y \in Q_{12} \setminus \{e\}$, $\langle e, y \rangle = \langle y \rangle$ and gcd$(|e|, |y|) = 1$. Additionally, $\langle a^2, a^3 \rangle = \langle a^3, a^4 \rangle = \langle a \rangle$, and gcd$(|a^2|, |a^3|) =$ gcd$(|a^3|, |a^4|) = 1$. Therefore, the graph $\Gamma^{epc}(Q_{12})$ can be represented as shown in Figure 3 and expressed as $\Gamma^{epc}(Q_{12}) = K_1 + (\overline{K}_8 \cup K_{1,2})$.*

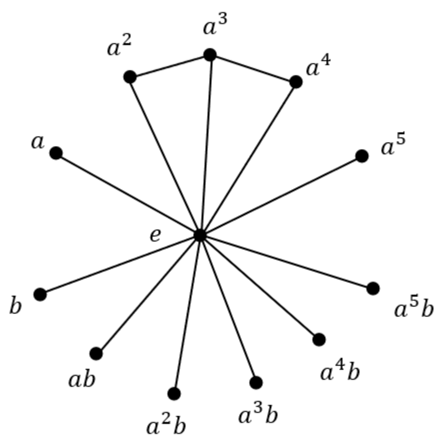

Figure 3: The enhanced power coprime graph of Q_{12}.

Adjacency information and element orders are also provided to support the proof of the general presentation of the enhanced power coprime graph for dihedral, semi-dihedral, and generalized quaternion groups.

Proposition 3.1. *Consider G as one of the groups D_n, SD_{2^n}, or Q_{4n}. For all $a^{m_1}, a^{m_2}, a^{m_3} \in G$, where $0 \leq m_1, m_2, m_3 < \dfrac{|G|}{2}$ and are isomorphic to the rotation elements, $\langle a^{m_1}, a^{m_2} \rangle = \langle a^{m_3} \rangle$ if and only if $m_1 \neq m_2$ and gcd$(m_1, m_2) = m_3$.*

Proof. Consider G as one of the groups D_n, SD_{2^n}, or Q_{4n}. Let $a^{m_1}, a^{m_2}, a^{m_3} \in G$, where $0 \leq m_1, m_2, m_3 < \frac{|G|}{2}$ and are isomorphic to the rotation elements in $\langle a \rangle \subset G$. If $m_1 \neq m_2$ and $\gcd(m_1, m_2) = m_3$, where m_3 can be equal to m_1 or m_2, it follows that m_3 divides both m_1 and m_2. Now, $a^{m_1} = (a^{m_3})^{\frac{m_1}{m_3}}$ and $a^{m_2} = (a^{m_3})^{\frac{m_2}{m_3}}$, which implies that a^{m_1} and a^{m_2} are powers of a^{m_3}. Hence, $\langle a^{m_1}, a^{m_2} \rangle$ is the subgroup generated by a^{m_1} and a^{m_2}, which is contained within $\langle a^{m_3} \rangle$. Additionally, $\langle a^{m_3} \rangle$ is a cyclic subgroup and contains both a^{m_1} and a^{m_2} because they are powers of a^{m_3}. Since $\langle a^{m_1}, a^{m_2} \rangle$ is contained in $\langle a^{m_3} \rangle$ and $\langle a^{m_3} \rangle$ contains $\langle a^{m_1}, a^{m_2} \rangle$, it follows that $\langle a^{m_1}, a^{m_2} \rangle = \langle a^{m_3} \rangle$. □

Proposition 3.2. *Let D_n be the dihedral group of order $2n$. For all integers i with $0 \leq i < n$, $|a^i| = \frac{n}{gcd(i,n)}$ and $|a^i b| = 2$.*

Proof. Suppose $D_n = \langle a \rangle \cup \{a^i b \mid 0 \leq i < n\}$, where $\langle a \rangle = \{e, a, a^2, \ldots, a^{n-1}\}$. Since $a^i \in \langle a \rangle$, we have $|a^i| = \frac{n}{\gcd(i,n)}$ according to the basic properties of the cyclic groups. Also, due to the relations $b^2 = e$ and $ba = a^{-1}b$ in D_n, $|a^i b| = 2$. □

Proposition 3.3. *Let D_n be the dihedral group of order $2n$. For all $x, y \in D_n$, $\langle x, y \rangle$ is equal to a proper cyclic subgroup of G if,*

1. $x, y \in \langle a \rangle$,
2. $x = e$ *and* $y = a^i b$,

where $0 \leq i < n$.

Proof. The result follows directly from Lemma 2.1, Remark 2.1, Propositions 3.1 and 3.2. □

Proposition 3.4. *Let Q_{4n} be the generalized quaternion group of order $4n$. For all integers k with $0 \leq k < 2n$, $|a^k| = \frac{2n}{gcd(k,2n)}$ and $|a^k b| = 4$.*

Proof. Suppose $Q_{4n} = \langle a \rangle \cup \{a^k b \mid 0 \leq k < 2n\}$, where $\langle a \rangle = \{e, a, a^2, \ldots, a^{2n-1}\}$. Using the basic properties of cyclic groups, $|a^k| = \frac{2n}{\gcd(k,2n)}$ for all $a^k \in \langle a \rangle$. Also, by the definition of Q_{4n}, $|a^k b| = 4$, due to the specific properties of the group. □

Proposition 3.5. *Let Q_{4n} be the generalized quaternion group of order $4n$. For all $x, y \in Q_{4n}$, $\langle x, y \rangle$ is equal to a proper cyclic subgroup of G if,*

1. $x, y \in \langle a \rangle$,
2. $x = e$ *and* $y = a^k b$,
3. $x = a^n$ *and* $y = a^k b$,
4. $x = a^k b$ *and* $y = a^{k+n} b$,

where $0 \leq k < 2n$.

Proof. The result follows directly from Lemma 2.1, Remark 2.1, Propositions 3.1 and 3.4. □

Proposition 3.6. *Consider G as one of the groups D_n, SD_{2^n}, or Q_{4n}. For all $x, y \in G$, gcd$(|x|, |y|) = 1$ if and only if,*

1. *either $|x| = 1$ or $|y| = 1$,*
2. *$|x| = g^r$ and $|y| = h^s$, where g, h are distinct prime powers and $r, s \in \mathbb{N}$.*

Proof. From Theorems 2.1 and 2.2, together with Propositions 3.2 and 3.4, it follows that if $|x| = 1$ or $|y| = 1$, and $|x| = g^r$ and $|y| = h^s$, where g, h are distinct prime powers and $r, s \in \mathbb{N}$, then $\gcd(|1|, |y|) = \gcd(|x|, |1|) = \gcd(|g^r|, |h^s|) = 1$. Otherwise, $\gcd(|x|, |y|) \neq 1$. □

Proposition 3.7. *Let G be one of the groups D_n, SD_{2^n}, or Q_{4n}. For all $x, y \in G$, $\langle x, y\rangle$ generates a proper cyclic subgroup of G and gcd$(|x|, |y|) = 1$ if and only if,*

1. *$x, y \in \langle a\rangle$, where $|x|$ and $|y|$ are distinct prime powers.*
2. *x is the identity element e, and $y \in G \setminus \{e\}$.*

Proof. Clearly, from Propositions 2.1, 3.3, 3.5, and 3.6, the result follows. □

The general presentations of the enhanced power coprime graph for certain cases of the mentioned groups are given in Theorems 3.1 and 3.2.

Theorem 3.1. *Let G be one of the groups D_n for $n = p^t$, where p is a prime number and $t \in \mathbb{N}$, or SD_{2^n}. Then, $\Gamma^{epc}(G) = K_{1,|G|-1}$.*

Proof. From Proposition 3.7, it follows that for $n = p^t$, where p is a prime number and $t \in \mathbb{N}$, $\langle x, y\rangle$ generates a proper cyclic subgroup of G and $\gcd(|x|, |y|) = 1$ if $x = e$ and $y \in G \setminus \{e\}$. For all $x, y \in \langle a\rangle$, $|x| = g^r$ and $|y| = g^s$, where g^r, g^s are powers of the same prime with different exponents $r, s \in \mathbb{N}$. Therefore, $\Gamma^{epc}(G) = K_{|e|,|G\setminus\{e\}|} = K_{1,|G|-1}$. This presentation corresponds to a star graph, where the central vertex e connects to all other vertices. □

Theorem 3.2. *Let Q_{4n} be the generalized quaternion group of order $4n$, where $n \geq 2$. Then, for $n = p^t$, where p is a prime number and $t \in \mathbb{N}$,*

$$\Gamma^{epc}(Q_{4n}) = \begin{cases} K_{1,4n-1}, & \text{if} \quad p = 2, \\ K_1 + (\overline{K}_{3n-1} \cup K_{1,n-1}), & \text{if} \quad p \neq 2. \end{cases}$$

Proof. Let $Q_{4n} = \langle a\rangle \cup \{a^k b \mid 0 \leq k < 2n\}$, where $\langle a\rangle = \{e, a, a^2, \ldots, a^{2n-1}\}$, and let $n = p^t$, where p is a prime number and $t \in \mathbb{N}$. Using Proposition 3.7, two cases need to be considered:

Case 1: If $p = 2$, the order of each element is 2^w, where $w \in \mathbb{N}$, for all $x, y \in Q_{4n} \setminus \{e\}$, and $\gcd(|x|, |y|) \neq 1$. Thus, $x \sim y$ only if $x = e$ and $y \in Q_{4n} \setminus \{e\}$. Therefore,

$$\Gamma^{epc}(Q_{4n}) = K_{|e|,|Q_{4n}\setminus\{e\}|} = K_{1,4n-1},$$

which corresponds to a star graph.

Case 2: If $p \neq 2$, $\Gamma^{epc}(Q_{4n}) = K_{|e|} + (\overline{K}_J \cup K_M)$, where J is the set of isolated vertices in the graph, and M is the set of vertices decomposed into two disjoint sets, a^n and $\bigcup_{k=1}^{n-1}\{a^{2k}\}$, for all $k \in \mathbb{N}$, where $\gcd(|x|, |y|) = 1$. Hence,

$$\Gamma^{epc}(Q_{4n}) = K_1 + (\overline{K}_{3n-1} \cup K_{1,n-1}).$$

□

3.2 Some characteristics of the graphs of groups

First, the vertex degrees of $\Gamma^{epc}(G)$ in specific cases of dihedral groups, semi-dihedral groups, and generalized quaternion groups are presented in Proposition 3.8.

Proposition 3.8. *Let $\Gamma^{epc}(G)$ be the enhanced power coprime graph of a finite group G, where G is SD_{2^n} or the prime power cases of D_n and Q_{4n}. For all $0 \leq m < \frac{|G|}{2}$,*

1. $deg_{\Gamma^{epc}(G)}(e) = |G| - 1$,
2. $deg_{\Gamma^{epc}(G)}(a^m) = \begin{cases} n, & \text{if} \quad m = n, \quad \text{where} \quad n = p^t, \quad p \neq 2 \text{ in } Q_{4n}, \\ 2, & \text{if} \quad m = 2\mu, \quad \text{where} \quad \mu < n, \quad \text{for} \quad n = p^t, \quad p \neq 2 \text{ in } Q_{4n}, \\ 1, & \text{otherwise}, \end{cases}$
3. $deg_{\Gamma^{epc}(G)}(a^m b) = 1$,

where p is a prime number, and $t, \mu \in \mathbb{N}$.

Proof. Let $\Gamma^{epc}(G)$ be the enhanced power coprime graph of a finite group G, where G is SD_{2^n} or the prime power cases of D_n and Q_{4n}. From Theorems 3.1 and 3.2, it is clear that $e \in V(\Gamma^{epc}(G))$ is the central vertex always connected to all other vertices. Therefore, $\deg_{\Gamma^{epc}(G)}(e) = |G| - 1$. For Q_{4n}, where $n = p^t$, $p \neq 2$, and $t \in \mathbb{N}$, $\Gamma^{epc}(G) = K_1 + (\overline{K}_{3n-1} \cup K_{1,n-1})$.

For all $a^m \in V(\overline{K}_{3n-1} \cup K_{1,n-1})$, two cases need to be considered:

Case 1: If $m = n$, then a^n is connected to e and $\bigcup_{m=1}^{n-1}\{a^{2m}\}$, where $m \in \mathbb{N}$ and $\gcd(x, y) = 1$. Hence, $\deg_{\Gamma^{epc}(G)}(a^m) = n$.

Case 2: If $m = 2\mu$, where $\mu < n$ and $\mu \in \mathbb{N}$, $a^{2\mu}$ is connected to a^n and e. Therefore, $\deg_{\Gamma^{epc}(G)}(a^m) = 2$.

Otherwise, $\deg_{\Gamma^{epc}(G)}(x) = 1$, since $x \in V(\Gamma^{epc}(G))$ is only connected to the central vertex e, including all $a^m b \in G$. □

In the following, Propositions 3.9 and 3.10 provide the results for the clique number, the chromatic number, the independence number, the domination number, the girth, and the diameter of the enhanced power coprime graph for certain cases of the mentioned groups, offering a thorough understanding of their relationships.

Proposition 3.9. *Let $\Gamma^{epc}(G)$ be the enhanced power coprime graph of the group G, where G is one of the groups D_n if $n = p^t$, SD_{2^n} for all n, or Q_{4n} if $n = 2^t$, where p is a prime number and $t \in \mathbb{N}$. Then,*

1. $\omega(\Gamma^{epc}(G)) = \chi(\Gamma^{epc}(G)) = diam(\Gamma^{epc}(G)) = 2$,
2. $\alpha(\Gamma^{epc}(G)) = |G| - 1$,
3. $\gamma(\Gamma^{epc}(G)) = 1$,
4. $girth(\Gamma^{epc}(G)) = \infty$.

Proof. From Theorems 3.1 and 3.2, it is clear that $\Gamma^{epc}(G) = K_{1,|G|-1}$, where G is one of the groups D_n if $n = p^t$, SD_{2^n} for all n, or Q_{4n} if $n = 2^t$, where p is a prime number and $t \in \mathbb{N}$. The graph invariants can be proved as follows,

1. The central vertex and any single leaf vertex form the largest complete subgraph in $K_{1,|G|-1}$, requiring only two colors for proper vertex coloring. Therefore,
$$\omega(\Gamma^{epc}(G)) = \chi(\Gamma^{epc}(G)) = 2.$$
Additionally, the maximum shortest distance between any two leaf vertices is 2, as it passes through the central vertex. Thus, diam$(\Gamma^{epc}(G)) = 2$.
2. The largest independent set consists of the leaf vertices. Thus, $\alpha(\Gamma^{epc}(G)) = |G| - 1$.
3. All vertices in the graph $V(K_{1,|G|-1})$ are directly connected to the central vertex. Hence, the smallest dominating set consists of only the central vertex, and $\gamma(\Gamma^{epc}(G)) = 1$.
4. Since $\Gamma^{epc}(G) = K_{1,|G|-1}$ has no cycles, its girth is infinite.

□

Proposition 3.10. *Consider $\Gamma^{epc}(Q_{4n})$ as the enhanced power coprime graph of the generalized quaternion group. Let p be a prime number with $p \neq 2$ and $n = p^t$ where $t \in \mathbb{N}$. Then,*

1. $\omega(\Gamma^{epc}(Q_{4n})) = \chi(\Gamma^{epc}(Q_{4n})) = girth(\Gamma^{epc}(Q_{4n})) = 3$,
2. $diam(\Gamma^{epc}(Q_{4n})) = 2$,
3. $\alpha(\Gamma^{epc}(Q_{4n})) = 4n - 2$,
4. $\gamma(\Gamma^{epc}(Q_{4n})) = 1$.

Proof. From Theorem 3.2, we have $\Gamma^{epc}(Q_{4n}) = K_1 + (\overline{K}_{3n-1} \cup K_{1,n-1})$ for $n = p^t$ with $p \neq 2$ and $t \in \mathbb{N}$. Now,

1. The join operation $K_1 + K_{1,n-1}$ forms a maximum complete subgraph K_3 using the vertex $v \in K_1$ and two vertices from $K_{1,n-1}$. Therefore, $\omega(\Gamma^{epc}(Q_{4n})) = 3$. Additionally, since the maximum component can be colored with three colors, the chromatic number $\chi(\Gamma^{epc}(Q_{4n})) = 3$. The presence of a triangle in the subgraph K_3 implies that the smallest cycle length is 3. Hence, girth$(\Gamma^{epc}(Q_{4n})) = 3$.
2. Let $v \in V(K_1)$, $w \in V(\overline{K}_{3n-1})$, and $z \in V(K_{1,n-1})$. Since $v \sim w$ and $v \sim z$, but $w \nsim z$, we have ecc$(v) = 1$ and ecc$(w) =$ ecc$(z) = 2$. Thus, the maximum shortest distance between any two vertices involves w and z, passing through v, so diam$(\Gamma^{epc}(Q_{4n})) = 2$.

3. Let $A = \{q_1, q_2, \ldots, q_{3n-1}\}$ represent the vertices of $\overline{K}_{3n-1}$, and $B = \{s, u_1, u_2, \ldots, u_{n-1}\}$ represent the vertices of $K_{1,n-1}$. The set A is independent, as there are no edges among its vertices, and $B \setminus \{s\}$ forms another independent set. Combining these sets,
$$\{q_1, q_2, \ldots, q_{3n-1}, u_1, u_2, \ldots, u_{n-1}\},$$
gives the maximum independent set. Hence,
$$\alpha(\Gamma^{epc}(Q_{4n})) = |A| + |B \setminus \{s\}| = (3n-1) + (n-1) = 4n-2.$$
4. The vertex $v \in K_1$ dominates every other vertex in $\Gamma^{epc}(Q_{4n})$, as it is adjacent to all vertices in $V(\overline{K}_{3n-1}) \cup V(K_{1,n-1})$. Hence, the set $\{v\}$ forms a dominating set of size 1, and $\gamma(\Gamma^{epc}(Q_{4n})) = 1$.

□

The classifications of the enhanced power coprime graph in terms of connectivity, completeness, regularity, planarity, and perfectness for dihedral groups, semi-dihedral groups, and generalized quaternion groups are presented in Proposition 3.11.

Proposition 3.11. *Let G be one of the groups SD_{2^n} for all n, D_n, or Q_{4n}, where $n = p^t$, p is a prime number, and $t \in \mathbb{N}$. Then, $\Gamma^{epc}(G)$ is connected, planar, and perfect, but it is neither a complete nor a regular graph.*

Proof. From Theorems 3.1 and 3.2, $\Gamma^{epc}(G)$ is connected because every vertex is connected to the central vertex, ensuring a path between any pair of vertices. Also, the central vertex can be placed in the center, with all other vertices surrounding it and connected by non-crossing edges; therefore, $\Gamma^{epc}(G)$ is planar.

From Propositions 3.9 and 3.10, the largest clique size is 2, and the chromatic number is also 2, which means it can be colored with two colors. Every induced subgraph maintains this property, which confirms that $\Gamma^{epc}(G)$ is a perfect graph. Finally, as shown in Proposition 3.8, the central vertex has a degree of $|G| - 1$, while other vertices have varying degrees. Since some vertices are not adjacent to all others and the degrees are not uniform, $\Gamma^{epc}(G)$ is neither complete nor regular. □

The results related to the Laplacian spectrum are then explained. Using the general presentation of $\Gamma^{epc}(G)$, the Laplacian matrix is constructed. Then, the characteristic polynomial of the Laplacian matrix and the Laplacian spectrum of $\Gamma^{epc}(G)$ are derived.

Theorem 3.3. *Let D_n be the dihedral group of order $2n$, where $n = p^t$ and $n \geq 3$. Then, the characteristic polynomial,*
$$\Theta(\Gamma^{epc}(D_n)) = \rho(\rho - 2n)(\rho - 1)^{2(n-1)},$$
and the Laplacian spectrum,
$$L_{spec}(\Gamma^{epc}(D_n)) = \{2n, 1^{2n-2}, 0\}.$$

Proof. From Theorem 3.1, the Laplacian matrix $L(\Gamma^{epc}(D_n))$ is a $2n \times 2n$ matrix where the rows and columns are indexed by the elements of the dihedral group D_n of order $2n$, with $n = p^t$ and $n \geq 3$. The matrix is given by,

$$L(\Gamma^{epc}(D_n)) = \begin{pmatrix} 2n-1 & -1_{1\times(2n-1)} \\ -1_{(2n-1)\times 1} & I_{(2n-1)\times(2n-1)} \end{pmatrix}.$$

The characteristic polynomial is

$$\Theta(L(\Gamma^{epc}(D_n))) = \begin{vmatrix} \rho-(2n-1) & 1_{1\times(2n-1)} \\ 1_{(2n-1)\times 1} & (\rho-1)I_{(2n-1)\times(2n-1)} \end{vmatrix}.$$

Using elementary row operations $R_1 \to (\rho-1)R_1 - (R_2 + R_3 + \ldots + R_{2n})$, we obtain

$$\Theta(L(\Gamma^{epc}(D_n))) = \frac{(\rho-(2n-1))(\rho-1)-(2n-1)}{\rho-1} \cdot |(\rho-1)I_{(2n-1)\times(2n-1)}|.$$

Simplifying, we get

$$\Theta(L(\Gamma^{epc}(D_n))) = \rho(\rho-1)^{2n-2}(\rho-2n).$$

From (1), the Laplacian spectrum is $L_{spec}(\Gamma^{epc}(D_n)) = \{2n, 1^{2n-2}, 0\}$ for all $n = p^t$ with $n \geq 3$. ☐

By following the same steps as in the proof of Theorem 3.3, the Laplacian spectrum results for other cases involving the star graph, as shown in Theorems 3.1 and 3.2, are also valid. Furthermore, for Q_{4n}, when $n = p^t$ with $p \neq 2$ and $t \in \mathbb{N}$, the Laplacian spectrum result is presented in Theorem 3.4.

Proposition 3.12. *Let SD_{2^n} be the semi-dihedral group of order 2^n, where $n \geq 4$. Then, the characteristic polynomial is*

$$\Theta(\Gamma^{epc}(SD_{2^n})) = \rho(\rho-2^n)(\rho-1)^{2^n-2},$$

and the Laplacian spectrum is

$$L_{spec}(\Gamma^{epc}(SD_{2^n})) = \{2^n, 1^{2^n-2}, 0\}.$$

Proposition 3.13. *Let Q_{4n} be the generalized quaternion group of order $4n$, where $n \geq 2$ and $n = 2^t$ with $t \in \mathbb{N}$. Then, the characteristic polynomial is*

$$\Theta(\Gamma^{epc}(Q_{4n})) = \rho(\rho-4n)(\rho-1)^{4n-2},$$

and the Laplacian spectrum is

$$L_{spec}(\Gamma^{epc}(Q_{4n})) = \{4n, 1^{4n-2}, 0\}.$$

Theorem 3.4. *Let Q_{4n} be the generalized quaternion group of order $4n$ for $n \geq 2$ where $n = p^t$ with $p \neq 2$ and $t \in \mathbb{N}$. The characteristic polynomial,*

$$\Theta(\Gamma^{epc}(Q_{4n})) = \rho(\rho-1)^{3n-1}(\rho-2)^{n-2}(\rho-(n+1))(\rho-4n),$$

and the Laplacian spectrum of

$$\Gamma^{epc}(Q_{4n}) = \{4n, n+1, 2^{n-2}, 1^{3n-1}, 0\}.$$

Proof. The Laplacian matrix of $\Gamma^{epc}(Q_{4n})$, denoted as $L(\Gamma^{epc}(Q_{4n}))$, is a $4n \times 4n$ matrix. The rows and columns are ordered according to their degree of the vertices for the generalized quaternion group of order $4n$, Q_{4n}, for $n \geq 2$ where $n = p^t$, $p \neq 2$, and $t \in \mathbb{N}$,

$$L(\Gamma^{epc}(Q_{4n})) = \begin{pmatrix} 4n-1 & -1 & -1_{1\times n-1} & -1_{1\times 3n-1} \\ -1 & n & -1_{1\times n-1} & 0_{1\times 3n-1} \\ -1_{n-1\times 1} & -1_{n-1\times 1} & 2I_{n-1\times n-1} & 0_{n-1\times 3n-1} \\ -1_{3n-1\times 1} & 0_{3n-1\times 1} & 0_{3n-1\times n-1} & I_{3n-1\times 3n-1} \end{pmatrix},$$

Then, the characteristic polynomial

$$\Theta(L(\Gamma^{epc}(Q_{4n}))) = \begin{vmatrix} \rho-(4n-1) & 1 & 1_{1\times n-1} & 1_{1\times 3n-1} \\ 1 & \rho-n & 1_{1\times n-1} & 0_{1\times 3n-1} \\ 1_{n-1\times 1} & 1_{n-1\times 1} & (\rho-2)I_{n-1\times n-1} & 0_{n-1\times 3n-1} \\ 1_{3n-1\times 1} & 0_{3n-1\times 1} & 0_{3n-1\times n-1} & (\rho-1)I_{3n-1\times 3n-1} \end{vmatrix}. \tag{2}$$

Applying the row operation $R_1 \to (\rho-1)R_1 - (R_2 + R_3 + \ldots + R_{4n})$ on the matrix in (2). Note that,

$$(\rho-1)R_1 = ((\rho-1)(\rho-(4n-1)), \rho-1, \rho-1, \ldots, \rho-1),$$

and the sum of all rows R_2 to R_{4n} is

$$\sum_{i=2}^{4n} R_i = (4n-1, \rho-1, \rho-1, \rho-1, \rho-1, \rho-1, \ldots, \rho-1),$$

which transforms R_1 into

$$\begin{aligned} R_1' &= ((\rho-1)(\rho-(4n-1)) - (4n-1), 0, 0, 0, 0, 0, \ldots, 0) \\ &= (\rho(\rho-4n), 0, 0, 0, 0, 0, \ldots, 0). \end{aligned}$$

Now,

$$\Theta(L(\Gamma^{epc}(Q_{4n}))) = \frac{1}{\rho-1} \begin{vmatrix} \rho(\rho-4n) & 0 & 0_{1\times n-1} & 0_{1\times 3n-1} \\ 1 & \rho-n & 1_{1\times n-1} & 0_{1\times 3n-1} \\ 1_{n-1\times 1} & 1_{n-1\times 1} & (\rho-2)I_{n-1\times n-1} & 0_{n-1\times 3n-1} \\ 1_{3n-1\times 1} & 0_{3n-1\times 1} & 0_{3n-1\times n-1} & (\rho-1)I_{3n-1\times 3n-1} \end{vmatrix},$$

and it is equivalent to

$$\Theta(L(\Gamma^{epc}(Q_{4n}))) = \frac{\rho(\rho-4n)}{\rho-1} \begin{vmatrix} \rho-n & 1_{1\times n-1} & 0_{1\times 3n-1} \\ 1_{n-1\times 1} & (\rho-2)I_{n-1\times n-1} & 0_{n-1\times 3n-1} \\ 0_{3n-1\times 1} & 0_{3n-1\times n-1} & (\rho-1)I_{3n-1\times 3n-1} \end{vmatrix}. \tag{3}$$

Now, perform the row operation on the remaining matrix in (3),
$R_1 \to (\rho-2)R_1 - (R_2 + R_3 + \ldots + R_n)$ gives

$$\Theta(L(\Gamma^{epc}(Q_{4n}))) = \frac{\rho(\rho-4n)}{(\rho-1)(\rho-2)} \begin{vmatrix} (\rho-2)(\rho-n)-(n-1) & 0_{1\times n-1} & 0_{1\times 3n-1} \\ 1_{n-1\times 1} & (\rho-2)I_{n-1\times n-1} & 0_{n-1\times 3n-1} \\ 0_{3n-1\times 1} & 0_{3n-1\times n-1} & (\rho-1)I_{3n-1\times 3n-1} \end{vmatrix}.$$

Consequently,

$$\begin{aligned} \Theta(L(\Gamma^{epc}(Q_{4n}))) &= \frac{\rho(\rho-4n)((\rho-2)(\rho-n)-(n-1))}{(\rho-1)(\rho-2)} |(\rho-2)I_{n-1\times n-1}||(\rho-1)I_{3n-1\times 3n-1}| \\ &= \frac{\rho(\rho-4n)(\rho^2-(n+2)\rho+n+1)}{(\rho-1)(\rho-2)} (\rho-2)^{n-1}(\rho-1)^{3n-1}. \end{aligned}$$

Upon simplifying and rearranging, therefore,

$$\begin{aligned}\Theta(L(\Gamma^{epc}(Q_{4n}))) &= \frac{\rho(\rho-4n)(\rho-(n+1))(\rho-1)}{(\rho-1)(\rho-2)}(\rho-2)^{n-1}(\rho-1)^{3n-1}\\ &= \rho(\rho-1)^{3n-1}(\rho-2)^{n-2}(\rho-(n+1))(\rho-4n).\end{aligned}$$

From (1), the Laplacian spectrum $\Gamma^{epc}(Q_{4n}) = \{4n, n+1, 2^{n-2}, 1^{3n-1}, 0\}$ for $n = p^t$, $p \neq 2$, and $t \in \mathbb{N}$. $\square$

4 Conclusions and Recommendations

In conclusion, this paper introduced the enhanced power coprime graph, $\Gamma^{epc}(G)$, as a tool to understand group structures, generate patterns, and order properties. General presentations were established for all semi-dihedral groups and prime power cases of dihedral and generalized quaternion groups. These presentations facilitated a comprehensive analysis of key graph characteristics, including graph invariants, classifications, and Laplacian spectra. In all analyzed cases, the graphs were consistently classified as connected, planar, and perfect, demonstrating a uniform structure. The theoretical results were fully consistent with those computed using Maple, confirming the accuracy of our approach. This study bridges group properties with graph-theoretic concepts, contributing to a deeper understanding of finite group structures. These findings could be expanded by exploring additional characteristics derived from the general presentations or developing presentations for other group cases. Moreover, these graphs offer potential applications in computational group theory, network analysis, coding theory, and the exploration of combinatorial symmetry in physical and chemical systems, providing valuable directions for further exploration.

Acknowledgement The first author gratefully acknowledges the support of the Malaysian Ministry of Higher Education (MoHE) and Universiti Teknologi MARA (UiTM) through the Skim Latihan Akademik Bumiputera (SLAB) scholarship.

Conflicts of Interest The authors declare that there are no conflicts of interest.

References

[1] G. Aalipour, S. Akbari, P. J. Cameron, R. Nikandish & F. Shaveisi (2017). On the structure of the power graph and the enhanced power graph of a group. *The Electronic Journal of Combinatorics*, *24*(3), Article ID: P3.16. https://doi.org/10.37236/6497.

[2] A. Adhikari & S. Banerjee (2022). Prime coprime graph of a finite group. *Novi Sad Journal of Mathematics*, *52*(2), 41–59. https://doi.org/10.30755/NSJOM.11151.

[3] S. Banerjee (2021). Laplacian spectra of coprime graph of finite cyclic and dihedral groups. *Discrete Mathematics, Algorithms and Applications*, *13*(03), Article ID: 2150020. https://doi.org/10.1142/S1793830921500208.

[4] M. Bello, N. M. Mohd Ali, & N. Zulkifli (2020). A systematic approach to group properties using its geometric structure. *European Journal of Pure and Applied Mathematics*, *13*(1), 84–95. https://doi.org/10.29020/nybg.ejpam.v13i1.3587.

[5] M. Bello, N. M. Mohd Ali & S. I. Isah (2021). Graph coloring using commuting order product prime graph. *Journal of Mathematics and Computer Science*, *23*(2), 155–169. http://dx.doi.org/10.22436/jmcs.023.02.08.

[6] S. Bera (2018). On the intersection power graph of a finite group. *Electronic Journal of Graph Theory and Applications*, *6*(1), 178–189. https://dx.doi.org/10.5614/ejgta.2018.6.1.13.

[7] S. Bera & A. K. Bhuniya (2018). On enhanced power graphs of finite groups. *Journal of Algebra and Its Applications*, *17*(8), Article ID: 1850146. https://doi.org/10.1142/S0219498818501463.

[8] S. Bera, H. K. Dey & S. K. Mukherjee (2021). On the connectivity of enhanced power graphs of finite groups. *Graphs and Combinatorics*, *37*(2), 591–603. https://doi.org/10.1007/s00373-020-02267-5.

[9] K. A. Bhat & G. Sudhakara (2018). Commuting graphs and their generalized complements. *Malaysian Journal of Mathematical Sciences*, *12*(1), 63–84.

[10] W. Burnside (1911). *Theory of Groups of Finite Order*. Cambridge: The University Press, Cambridge.

[11] P. J. Cameron (2022). Graphs defined on groups. *International Journal of Group Theory*, *11*(2), 53–107. https://doi.org/10.22108/ijgt.2021.127679.1681.

[12] P. J. Cameron & B. Kuzma (2023). Between the enhanced power graph and the commuting graph. *Journal of Graph Theory*, *102*(2), 295–303. https://doi.org/10.1002/jgt.22871.

[13] S. Dalal & J. Kumar (2021). On enhanced power graphs of certain groups. *Discrete Mathematics, Algorithms and Applications*, *13*(01), Article ID: 2050099. https://doi.org/10.1142/S1793830920500998.

[14] H. R. Dorbidi (2016). A note on the coprime graph of a group. *International Journal of Group Theory*, *5*(4), 17–22. https://doi.org/10.22108/ijgt.2016.9125.

[15] L. A. Dupont, D. G. Mendoza & M. Rodriguez (2023). The rainbow connection number of the enhanced power graph of a finite group. *Electronic Journal of Graph Theory and Applications*, *11*(1), 235–244. https://dx.doi.org/10.5614/ejgta.2023.11.1.19.

[16] J. Harris, J. L. Hirst & M. Mossinghoff (2008). *Combinatorics and Graph Theory*. Undergraduate Texts in Mathematics. Springer-Verlag, New York. https://doi.org/10.1007/978-0-387-79711-3.

[17] X. L. Ma, H. Q. Wei & L. Y. Yang (2014). The coprime graph of a group. *International Journal of Group Theory*, *3*(3), 13–23. https://doi.org/10.22108/ijgt.2014.4363.

[18] X. Ma, A. Kelarev, Y. Lin & K. Wang (2022). A survey on enhanced power graphs of finite groups. *Electronic Journal of Graph Theory and Applications*, *10*(1), 89–111. https://dx.doi.org/10.5614/ejgta.2022.10.1.6.

[19] N. Mohamed, N. M. Mohd Ali, & M. Bello (2024). Laplacian spectrum and energy of the cyclic order product prime graph of semi-dihedral groups. *Malaysian Journal of Fundamental and Applied Sciences*, *20*(3), 669–677. https://doi.org/10.11113/mjfas.v20n3.3422.

[20] N. Nurhabibah, I. G. A. W. Wardhana & N. W. Switrayni (2023). Numerical invariants of coprime graph of a generalized quaternion group. *Journal of The Indonesian Mathematical Society*, *29*(1), 36–44. https://doi.org/10.22342/jims.29.1.1245.36-44.

[21] R. P. Panda, S. Dalal & J. Kumar (2021). On the enhanced power graph of a finite group. *Communications in Algebra*, *49*(4), 1697–1716. https://doi.org/10.1080/00927872.2020.1847289.

[22] Parveen, S. Dalal & J. Kumar (2024). Enhanced power graphs of certain non-abelian groups. *Discrete Mathematics, Algorithms and Applications*, *16*(5), Article ID: 2350063. https://doi.org/10.1142/S1793830923500635.

[23] M. U. Romdhini, A. Nawawi & C. Y. Chen (2023). Neighbors degree sum energy of commuting and non-commuting graphs for dihedral groups. *Malaysian Journal of Mathematical Sciences*, *17*(1), 53–65. https://doi.org/10.47836/mjms.17.1.05.

[24] J. Schaefer & K. Schlechtweg (2017). On the structure of symmetric spaces of semidihedral groups. *Involve, a Journal of Mathematics*, *10*(4), 665–676. https://doi.org/10.2140/involve.2017.10.665.

[25] A. Sehgal, Manjeet & D. Singh (2021). Co-prime order graphs of finite Abelian groups and dihedral groups. *Journal of Mathematics and Computer Science*, *23*(3), 196–202. http://dx.doi.org/10.22436/jmcs.023.03.03.

[26] K. Selvakumar & M. Subajini (2017). Classification of groups with toroidal coprime graphs. *The Australasian Journal of Combinatorics*, *69*(2), 174–183.

[27] D. B. West (2001). *Introduction to Graph Theory*. Pearson Prentice Hall, Ann Arbor, Michigan.